Amalgamated Worksheet # 1

Various Artists

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Problem 1:

Find $T \in \mathcal{L}(\mathbb{R}^3)$ whose characteristic polynomial and minimal polynomial are not the same

Problem 2:

Find all 2×2 matrices A such that $A^2 - 3A + 2I = 0$

Hint: You may use the following result: If $\mathbb{R}^2$ has a basis of eigenvectors of A, then there exists a matrix P such that $A = PDP^{-1}$, where D is the matrix of eigenvalues of A

Problem 3:

If $dim(V) = n < \infty$, and $T \in \mathcal{L}(V)$ show that:

$$dim\left(Span\left\{I, T, T^2, \cdots\right\}\right) < n$$

Problem 4:

Find a formula for T^{-1} in terms of the coefficients of the characteristic polynomial of T

Problem 5:

(if time permits) Given $v \in V$, a polynomial g is called the $T-$**annihilator of** v (or $T-$*killer* of v) if $g(t)$ is a monic polynomial of *least* degree such that $g(T)v = 0$.

Show that such a g divides the minimal polynomial q of T

Problem 6:

(if time permits) Find an (infinite-dimensional) vector space V and a linear operator $D \in \mathcal{L}(V)$ with no minimal polynomial.

2 Daniel Sparks

Cayley-Hamilton via Matrix Multiplication and Jordan Form

Let T be a nilpotent operator, upper triangular with respect to the basis $\beta = v_1, \cdots, v_n$. Let $V_i = \text{Span}\{v_1, \cdots, v_i\}$ for $i = 1, \cdots, n$, and let $V_0 = (0)$.

(a) Prove that $T^i(V_i) = (0)$. What does this say about the matrix of T^i with respect to β?

(b) Prove that $T^i(V_j) \subset V_{j-i}$, so long as $j - i \geq 0$. What does this say about the matrix of T^i with respect to β?

(c) Let M be a strictly upper-triangular, $n \times n$ matrix. By the "first diagonal" I shall mean the main diagonal, entries $M_{1,1}$ through $M_{n,n}$. By the "second diagonal" I shall mean the entries above these ones, i.e. $M_{1,2}, \cdots, M_{n-1,n}$. (So the n-th diagonal is just the upper right entry, $M_{1,n}$.)

Prove that, in the matrix M^i, the entries on or below the i-th diagonal are zero.

(d) Let $A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_k \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix}$ be *block diagonal* matrices of the same dimension. (That is, A_i and B_i are square matrices of the same size.) Prove that

$$AB = \begin{pmatrix} A_1B_1 & 0 & \cdots & 0 \\ 0 & A_2B_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_kB_k \end{pmatrix}$$

[Hint: Induction, the base case being $k = 2$. For the inductive step $k = n \Rightarrow k = n+1$, you will use not only the result for $k = n$, but also for $k = 2$ again. (Parenthetical remark: this is an example of "strong" mathematical induction.)]

(e) Let L be a linear operator on W, a complex vector space of dimension n. Let $\lambda_1, \cdots, \lambda_m$ be the distinct eigenvalues of L. Let β be a basis for which L is in Jordan-Canonical form, and let $W = U_1 \oplus \cdots \oplus U_m$ be the corresponding decomposition into generalized eigenspaces. Writing $e_i = \dim U_i$, recall that the characteristic polynomial of L is defined to be

$$(T - \lambda_1)^{e_1} \cdots (T - \lambda_m)^{e_m}$$

Using the ideas discussed above, prove the Cayley-Hamilton theorem. That is, show that $(L - \lambda_1)^{e_1} \cdots (L - \lambda_m)^{e_m} = 0$.

[Hint: For simplicity, use the version of Jordan form which has one block per generalized eigenspace. Use parts (c) and (d) above.]